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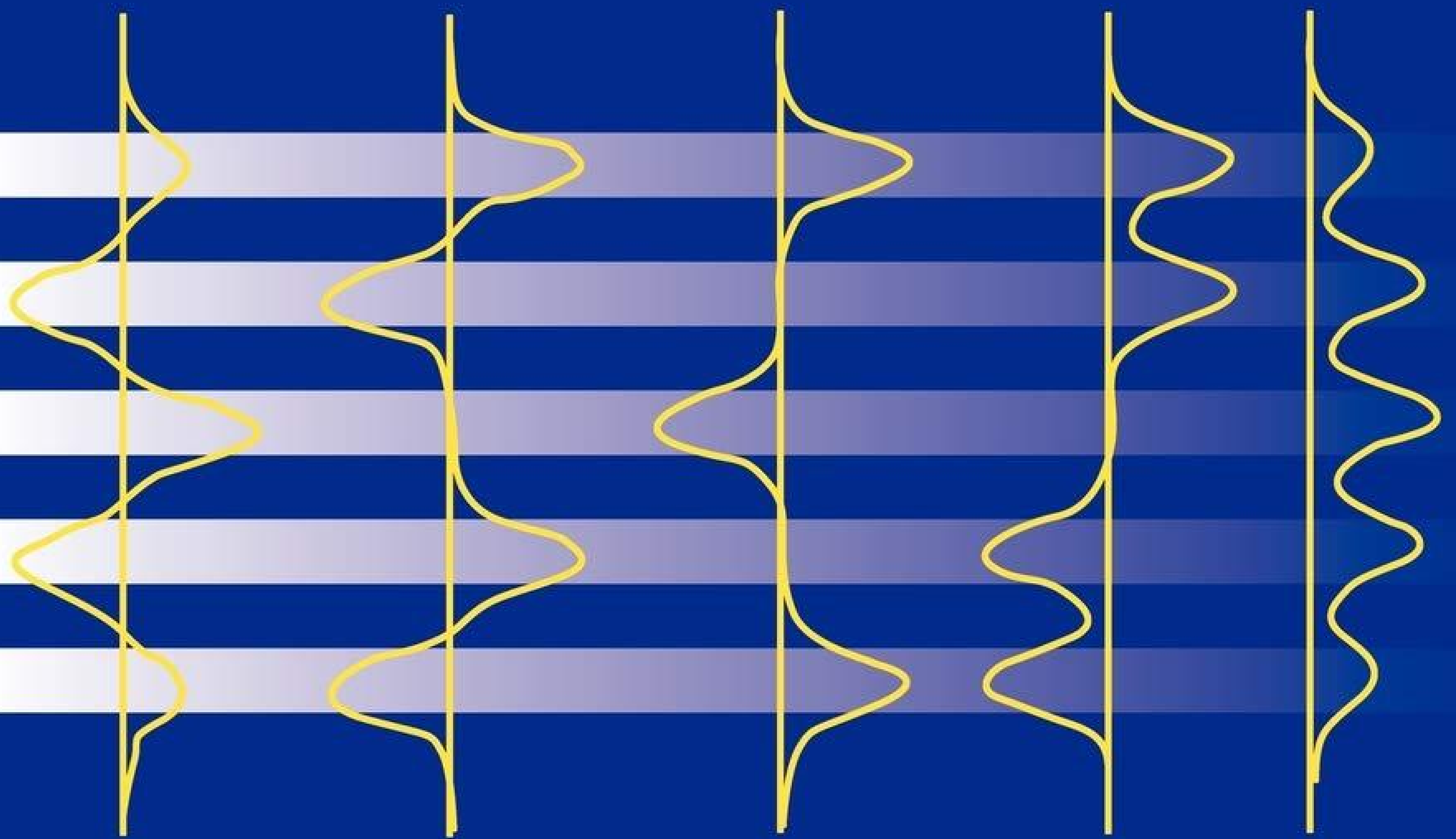

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# Optical Waves in Layered Media

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**Rockwell International Science Center, Thousand Oaks, California**

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# Preface

This book treats the theory of electromagnetic propagation in layered media. It is intended as a text for a course in modern optics for electrical engineering or applied physics students. Students are assumed to have general knowledge in electromagnetism and elementary matrix algebra. Some mathematical background in Fourier expansion and elementary differential equations would be helpful. The primary objectives of this book are to present a clear picture of the propagation of optical waves in layered media and to teach the reader how to analyze and design optical devices using such layered media. Although there exist a number of books on similar subjects, most are either too narrow in scope or are in the form of monographs, which are not suitable as textbooks.

Layered media play a very important role in many applications of modern optics. To fully utilize these media for transmission optics, we must understand the propagation of electromagnetic waves in these media. In addition, we must also be familiar with the systematic approaches used in the design of the layered structures. The emphasis is therefore on the theory of the propagation of optical waves in these media. An effort is made to bridge the gap between theory and practice through the use of numerical examples based on real situations. Only classical electrodynamics is used in dealing with the interaction of light with matter, except in the last chapter, where the subject of quantum wells is treated. Layered media that consist of isotropic and anisotropic materials are considered. Transmission and reflection of optical waves as well as the propagation of confined electromagnetic radiation are covered. A very wide range of topics is included, as may be seen from the table of contents.

I am deeply indebted to Professor Amnon Yariv for introducing the optics of layered media to me during the years when I was a graduate student at Caltech, and for his enlightening teaching. Portions of chapters 6, 9, and 11 first appeared, in different form, in *Optical Waves in Crystals*, co-authored by Professor Yariv and me. These materials are included here for completeness. I thank John Wiley & Sons and Professor Amnon Yariv for permission to reproduce these materials. My grateful thanks are also due to Drs. Joseph Longo, Derek Cheung, and Monte Khoshnevisan for their constant support and encouragement. I am also indebted to Drs. William Southwell and Kuo-Liang Chen and Mr. Paul Beckwith for their patient reading of the manuscript and helpful suggestions and to Sandy Nestor for her patient and competent typing of the manuscript. Finally, I am deeply grateful to my wife, Linda. Her love and devotion as a mother and wife have made the task at hand palatable and worthwhile.

POCHI YEH

*Thousand Oaks, California*
*May 1988*

# Contents

# 1

# The Electromagnetic Field

This book deals with the propagation of optical waves in layered media. In the first chapter, we review some of the basic properties of the propagation of electromagnetic radiation. These background materials are used frequently throughout the book and are included for completeness and as a ready source of reference.

We begin by briefly reviewing Maxwell's equations and the material equations. We then discuss the boundary conditions and the energy flow associated with electromagnetic radiation. These are followed by a derivation of the wave equations and an analysis of the propagation of monochromatic plane waves and some of their properties. Finally, we discuss the polarization state as well as the coherence of electromagnetic radiation.

## 1.1 MAXWELL'S EQUATIONS AND BOUNDARY CONDITIONS

### 1.1.1 Maxwell's Equations

The most fundamental equations in electrodynamics are Maxwell's equations, which are given in the following in rationalized MKS units:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.1-1}$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{1.1-2}$$

$$\nabla \cdot \mathbf{D} = \varrho, \tag{1.1-3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.1-4}$$

In these equations, **E** and **H** are the electric field vector (in volts per meter) and magnetic field vector (in amperes per meter), respectively. These two field vectors are often used to describe an electromagnetic field. The quantities **D** and **B** are called the electric displacement (in coulombs per square meter) and the magnetic induction (in webers per square meter), respectively. These two quantities are introduced to include the effect of the field on matter. The quantities $\varrho$ and $\mathbf{J}$ are the electric charge (in coulombs per cubic meter) and current (in amperes per square meter) densities, respectively, and may be considered as the sources of the fields **E** and **H**. These four Maxwell equations completely determine the electromagnetic field and are the fundamental equations of the theory of such fields, that is, of electrodynamics.

In optics, one often deals with propagation of electromagnetic radiation in regions of space where both charge density and current density are zero. In fact, if we set $\varrho = 0$ and $\mathbf{J} = 0$ in Maxwell's equations, we find that nonzero solutions exist. This means that an electromagnetic field can exist even in the absence of any charges and currents. Electromagnetic fields occurring in media in the absence of charges are called electromagnetic waves.

Maxwell's Equations (Eq. 1.1-1 to 1.1-4) consist of 8 scalar equations that relate a total of 12 variables, 3 for each of the 4 vectors **E**, **H**, **D**, and **B**. They cannot be solved uniquely unless the relationship between **B** and **H** and that between **E** and **D** are known. To obtain a unique determination of the field vectors, Maxwell's equations must be supplemented by the so-called constitutive equations (or material equations),

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{1.1-5}$$

$$\mathbf{B} = \mu\mathbf{H} = \mu_0\mathbf{H} + \mathbf{M}, \tag{1.1-6}$$

where the constitutive parameters $\varepsilon$ and $\mu$ are tensors of rank 2 and are known as the dielectric tensor (or permittivity tensor) and the permeability tensor, respectively; **P** and **M** are electric and magnetic polarizations, respectively. When an electromagnetic field is present in matter, the electric field can perturb the motion of electrons and produce a dipole polarization **P** per unit volume. Analogously, the magnetic field can also induce a magnetization **M** in materials having a permeability that is different from $\mu_0$. The constant $\varepsilon_0$ is called the permittivity of a vacuum and has a value of $8.854 \times 10^{-12}$ F/m. The constant $\mu_0$ is known as the permeability of a vacuum. It has, by definition, the exact value of $4\pi \times 10^{-7}$ H/m. If the material medium is isotropic, both $\varepsilon$ and $\mu$ tensors reduce to scalars. In many cases, the quantities $\varepsilon$ and $\mu$ can be assumed to be independent of the field strengths. However, if the fields are sufficiently strong, such as obtained, for example, by focusing a laser beam or applying a strong dc electric field to an electro-optic crystal, the dependence of these quantities on **E** and **H** must be considered. These nonlinear optical effects are beyond the scope of this book.

### 1.1.2 Boundary Conditions

One of the most important problems in determining the reflection and transmission of electromagnetic radiation through a layered medium is the continuity of some components of the field vectors at the dielectric interfaces between the layers. Although the physical properties (characterized by $\varepsilon$ and $\mu$) may change abruptly across the dielectric interfaces, there exist continuity relationships of some of the components of the field vectors at the dielectric boundary. These continuity conditions can be derived directly from Maxwell's equations.

Consider a boundary surface separating two media with different dielectric permittivity and permeability (medium 1 and medium 2). To obtain the boundary conditions for **B** and **D**, we construct a thin cylinder over a unit area of the surface, as shown in Fig. 1.1(*a*). The end faces of the cylinder are parallel to the surface. We now apply the Gauss divergence theorem

$$\int \nabla \cdot \mathbf{F}\, dV = \int \mathbf{F} \cdot d\mathbf{S} \tag{1.1-7}$$

to both sides of Eqs. (1.1-3) and (1.1-4). The surface integral reduces, in the limit as the height of the cylinder approaches zero, to an integral over the end surfaces only. This leads to

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0, \qquad \mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma, \tag{1.1-8}$$

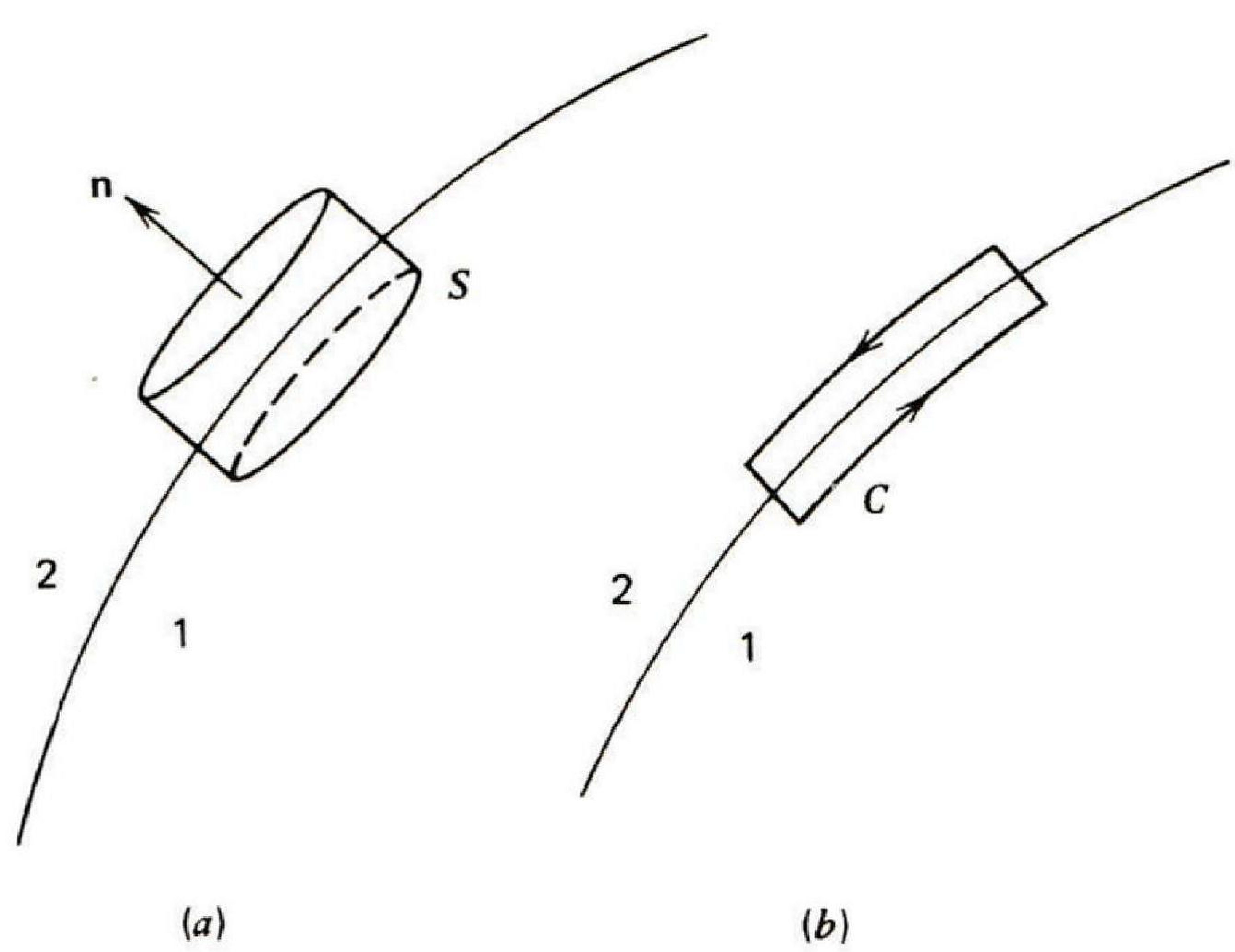


**Figure 1.1** (*a*) A short cylinder about the interface between two media: *S* is the surface of this cylinder. (*b*) A narrow rectangle about the interface between two media; *C* is the boundary of this rectangle. (Adapted from A. Yariv and P. Yeh, *Optical Waves in Crystals*, Wiley, New York, 1984, p. 3. Copyright © 1984. By permission of John Wiley & Sons, Inc.)

where $\mathbf{n}$ is the unit normal to the surface directed from medium 1 into medium 2, $\sigma$ is the surface charge density (in coulombs per square meter), and the subscripts refer to values at the surfaces in the two media. The boundary conditions (1.1-8) are often written as

$$B_{2n} = B_{1n}, \qquad D_{2n} - D_{1n} = \sigma, \tag{1.1-9}$$

where $B_{2n} = \mathbf{B}_2 \cdot \mathbf{n}$, $B_{1n} = \mathbf{B}_1 \cdot \mathbf{n}$, $D_{2n} = \mathbf{D}_2 \cdot \mathbf{n}$, and $D_{1n} = \mathbf{D}_1 \cdot \mathbf{n}$. In other words, the normal component of the magnetic induction $\mathbf{B}$ is always continuous, and the difference between the normal components of the electric displacement $\mathbf{D}$ is equal in magnitude to the surface charge density $\sigma$.

For the field vectors $\mathbf{E}$ and $\mathbf{H}$, we draw a rectangular contour with two long sides parallel to the surface of discontinuity, as shown in Fig. 1.1(*b*). We now apply the Stokes theorem

$$\int \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int \mathbf{F} \cdot dl \tag{1.1-10}$$

to both sides of Eqs. (1.1-1) and (1.1-2). The contour integral reduced, in the limit as the width of the rectangle approaches zero, to an integral over the two long sides only. This leads to

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \qquad \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}, \tag{1.1-11}$$

where $\mathbf{K}$ is the surface current density (in amperes per meter). Again, the boundary conditions for the electric and magnetic field vectors (1.1-11) are often written as

$$\mathbf{E}_{2t} = \mathbf{E}_{1t}, \qquad \mathbf{H}_{2t} - \mathbf{H}_{1t} = \mathbf{K}, \tag{1.1-12}$$

where the subscript $t$ means the tangential component of the field vector. (*Note:* The tangential components of these field vectors to the boundary surface are still vectors in the tangential plane of the surface.) In other words, the tangential component of the electric field vector $\mathbf{E}$ is always continuous at the boundary surface, and the difference between the tangential components of the magnetic field vector $\mathbf{H}$ is equal to the surface current density $\mathbf{K}$.

In many areas of optics, one often deals with situations in which the surface charge density $\sigma$ and the surface current density $\mathbf{K}$ both vanish. It follows that, in such a case, the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous across the interface separating media 1 and 2. These boundary conditions are important in solving many wave propagation problems in optics, such as guided-wave optics (Chapter 11) and wave propagation in layered media.

## 1.2 ENERGY DENSITY AND ENERGY FLUX

It has been known for some time that light carries energy with it and is a form of electromagnetic radiation. The first and most conspicuous success of Maxwell's theory was the prediction of the existence of electromagnetic waves and the transport of energy. We now consider two of the most important aspects of electrodynamics: the energy density stored with an electromagnetic wave and the energy flux associated with an electromagnetic wave. To derive the energy density and the energy flux, we consider the conservation of energy in a small volume. The work done per unit volume by an electromagnetic field is $\mathbf{J} \cdot \mathbf{E}$, which may also be considered as the energy dissipation per unit volume. This energy dissipation must be connected with the net decrease in the energy density and the energy flow out of the volume. According to Eq. (1.1-2), the work done by the electromagnetic field can be written as

$$\mathbf{J} \cdot \mathbf{E} = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{1.2-1}$$

If we now employ the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{1.2-2}$$

and use Eq. (1.1-1), the right side of (1.2-1) becomes

$$\mathbf{J} \cdot \mathbf{E} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}. \tag{1.2-3}$$

If we now further assume that the material medium involved is linear in its electromagnetic properties (i.e., $\varepsilon$ and $\mu$ are independent of the field strengths), Eq. (1.2-3) can be written as

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \tag{1.2-4}$$

where $U$ and $\mathbf{S}$ are defined as

$$U = \tfrac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}), \tag{1.2-5}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{1.2-6}$$

The scalar $U$ represents the energy density of the electromagnetic fields and has the dimensions of joules per cubic meter. The vector $\mathbf{S}$, representing the energy flux, is called Poynting's vector and has the dimensions of joules per square meter per second. It is consistent to view $|\mathbf{S}|$ as the power per unit area (watts per square meter) carried by the field in the direction of $\mathbf{S}$. The quantity $\nabla \cdot \mathbf{S}$ thus represents the net electromagnetic power flowing out of a unit volume. Equation (1.2-4) is known as the continuity equation or Poynting's theorem. It represents conservation of energy for the electromagnetic field. The conservation laws for the linear momentum of electromagnetic fields can be obtained in a similar way. This is left as a problem for the student (Problem 1.4).

## 1.3 COMPLEX NUMBERS AND MONOCHROMATIC FIELDS

It is known that monochromatic light has a unique angular frequency of oscillation. Although most light sources consist of a distribution of the angular frequencies, we will investigate the properties of layered media using monochromatic light. This is legitimate because throughout the book we assume that the materials involved in layered media are linear and that each frequency component of light interacts independently with the media. For monochromatic light, the field vectors are sinusoidal functions of time, and it is convenient to represent each field vector as a complex exponential function. The reason we do this is that it is easier to work with exponential functions than with cosine or sine. As an example, consider some component of the field vectors:

$$a(t) = |A| \cos(\omega t + \alpha), \tag{1.3-1}$$

where $\omega$ is the angular frequency and $\alpha$ is the phase. If we define a complex amplitude of $a(t)$ by

$$A = |A| e^{i\alpha}, \tag{1.3-2}$$

Eq. (1.3-1) can be written as

$$a(t) = \mathrm{Re}[A e^{i\omega t}]. \tag{1.3-3}$$

We will often represent $a(t)$ by

$$a(t) = A e^{i\omega t} \tag{1.3-4}$$

instead of by Eq. (1.3-1) or (1.3-3). This is sometimes referred to as the analytic representation. We must understand that the complex number so defined is not a real physical component because no electromagnetic field in physics is complex; actually, field vectors have no imaginary parts, only real parts. We shall, however, speak of the "field" $A \exp(i\omega t)$, but of course, the actual field is the real part of that expression. Using complex representations, the monochromatic fields are written as exponential functions. This leads to great mathematical simplification. For example, differentiation can now be replaced by a simple multiplication. The exceptions are cases that involve the product (or powers) of field vectors such as energy density and Poynting's vector. In these cases, one must use the real form of the physical quantities.

As an example, consider the product of two sinusoidal functions $a(t)$ and $b(t)$, where

$$\begin{aligned} a(t) &= |A|\cos(\omega t + \alpha) \\ &= \mathrm{Re}[Ae^{i\omega t}] \end{aligned} \tag{1.3-5}$$

and

$$\begin{aligned} b(t) &= |B|\cos(\omega t + \beta) \\ &= \mathrm{Re}[Be^{i\omega t}], \end{aligned} \tag{1.3-6}$$

with $A = |A|\exp(i\alpha)$ and $B = |B|\exp(i\beta)$. Using the real functions, we get

$$a(t)b(t) = \tfrac{1}{2}|AB|[\cos(2\omega t + \alpha + \beta) + \cos(\alpha - \beta)]. \tag{1.3-7}$$

But if we were to evaluate the product $a(t)b(t)$ with the complex form of the functions, we would get

$$a(t)b(t) = ABe^{i2\omega t} = |AB|e^{i(2\omega t + \alpha + \beta)}. \tag{1.3-8}$$

A comparison of the last result to Eq. (1.3-7) shows that the time-independent (dc) term $\frac{1}{2}|AB|\cos(\alpha - \beta)$ is missing, and thus the use of the complex form led to an error. Generally speaking, the product of the real part of two complex numbers may not be equal to the real part of the product of these two complex numbers. In other words, if $x$ and $y$ are two arbitrary complex numbers, the following is generally true:

$$\mathrm{Re}[x] \cdot \mathrm{Re}[y] \neq \mathrm{Re}[xy]. \tag{1.3-9}$$

### 1.3.1 Time Averaging of Sinusoidal Products

In optical fields, the field vectors are rapidly varying functions of time. For example, the period of a time-varying field with a wavelength $\lambda = 1\ \mu\text{m}$ is $T = \lambda/c = 0.33 \times 10^{-14}$ s. One often considers the time-averaged values rather than the instantaneous values of many physical quantities such as Poynting's vector and the energy density. It is frequently necessary to find the time average of the product of two sinusoidal functions of the same frequency:

$$\langle a(t)b(t)\rangle = \frac{1}{T}\int_0^T |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)\,dt, \quad (1.3\text{-}10)$$

where $a(t)$ and $b(t)$ are given by Eqs. (1.3-5) and (1.3-6) and the angle brackets denote time averaging; $T = 2\pi/\omega$ is the period of oscillation. Since the integral in Eq. (1.3-10) is periodic in $T$, the averaging can be performed over a time $T$. By using Eq. (1.3-7), we obtain directly

$$\langle a(t)b(t)\rangle = \tfrac{1}{2}|AB|\cos(\alpha - \beta) \quad (1.3\text{-}11)$$

since the average of $T$ of the term involving $\cos(2\omega t + \alpha + \beta)$ is zero. This last result can be written in terms of the complex amplitudes $A$ and $B$, defined immediately following Eq. (1.3-6) as

$$\langle a(t)b(t)\rangle = \tfrac{1}{2}\mathrm{Re}[AB^*] \quad (1.3\text{-}12)$$

or in terms of the analytic form of $a(t)$ and $b(t)$ directly as

$$\langle \mathrm{Re}[a(t)]\,\mathrm{Re}[b(t)]\rangle = \tfrac{1}{2}\mathrm{Re}[a(t)b^*(t)]. \quad (1.3\text{-}13)$$

where the superscript asterisk indicates the complex conjugate. The time dependence on the right side of Eq. (1.3-13) disappears because both $a(t)$ and $b(t)$ have the same sinusoidal time dependence $\exp(i\omega t)$. These two results, Eqs. (1.3-12) and (1.3-13), are important and will find frequent use throughout the book.

By using the complex formalism (or analytic representation) for the field vectors **E**, **H**, **D**, and **B**, the time-averaged Poynting's vector (1.2-6) and the energy density (1.2-5) for sinusoidally varying fields are given by

$$\mathbf{S} = \tfrac{1}{2}\mathrm{Re}[\mathbf{E} \times \mathbf{H}^*] \quad (1.3\text{-}14)$$

and

$$U = \tfrac{1}{4}\mathrm{Re}[\mathbf{E}\cdot\mathbf{D}^* + \mathbf{B}\cdot\mathbf{H}^*] \quad (1.3\text{-}15)$$

respectively.

## 1.4 WAVE EQUATIONS AND MONOCHROMATIC PLANE WAVES

Two of the most important results of Maxwell's equations are the wave equations and the existence of electromagnetic waves that are solutions to them. We now derive the wave equations in material media. This is achieved by mathematical elimination so that each of the field vectors satisfies a differential equation. We limit our attention to regions where both charge density $\varrho$ and current density $\mathbf{J}$ vanish. We also assume in this section that the medium is isotropic, so that $\varepsilon$ and $\mu$ are scalars.

If we use the constitutive relation (1.1-6) for $\mathbf{B}$ in Eq. (1.1-1), divide both sides by $\mu$, and apply the curl operator, we obtain

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \frac{\partial}{\partial t}\nabla \times \mathbf{H} = 0. \tag{1.4-1}$$

If we now differentiate Eq. (1.1-2) with respect to time, combine it with Eq. (1.4-1), and use the material Eq. (1.1-5), we obtain

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0. \tag{1.4-2}$$

We now employ the vector identities

$$\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) = \frac{1}{\mu}\nabla \times (\nabla \times \mathbf{E}) + \left(\nabla\frac{1}{\mu}\right) \times (\nabla \times \mathbf{E}) \tag{1.4-3}$$

and

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}, \tag{1.4-4}$$

and Eq. (1.4-2) becomes

$$\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + (\nabla \ln \mu) \times (\nabla \times \mathbf{E}) - \nabla(\nabla \cdot \mathbf{E}) = 0. \tag{1.4-5}$$

By substituting for $\mathbf{D}$ from Eq. (1.1-5) into Eq. (1.1-3) and applying the vector identity

$$\nabla \cdot (\varepsilon\mathbf{E}) = \varepsilon\nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla\varepsilon, \tag{1.4-6}$$

we obtain, from Eq. (1.4-5),

$$\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} + (\nabla \ln \mu) \times (\nabla \times \mathbf{E}) + \nabla(\mathbf{E} \cdot \nabla \ln \varepsilon) = 0. \tag{1.4-7}$$

This is the wave equation for the field vector $\mathbf{E}$. The wave equation for the magnetic field vector $\mathbf{H}$ can be obtained in a similar way and is given by

$$\nabla^2\mathbf{H} - \mu\varepsilon\frac{\partial^2 \mathbf{H}}{\partial t^2} + (\nabla \ln \varepsilon) \times (\nabla \times \mathbf{H}) + \nabla(\mathbf{H} \cdot \nabla \ln \mu) = 0. \tag{1.4-8}$$

Inside a homogeneous and isotropic medium, the gradient of the logarithm of $\varepsilon$ and $\mu$ vanishes, and the wave Eqs. (1.4-7) and (1.4-8) reduce to

$$\nabla^2\mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \nabla^2\mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \tag{1.4-9}$$

These are the standard electromagnetic wave equations. They are satisfied by the well-known monochromatic plane wave

$$\psi = A e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}, \tag{1.4-10}$$

where $A$ is a constant and is the amplitude. In Eq. (1.4-10), the angular frequency $\omega$ and the magnitude of the wave vector $\mathbf{k}$ are related by

$$|\mathbf{k}| = \omega\sqrt{\mu\varepsilon} \tag{1.4-11}$$

and $\psi$ can be any Cartesian component of $\mathbf{E}$ and $\mathbf{H}$.

Let us now examine the meaning of this solution. In each plane, $\mathbf{k}\cdot\mathbf{r}$ = constant (const), the field is a sinusoidal function of time. At each given moment, the field is a sinusoidal function of space. It is clear that the field has the same value for coordinates $\mathbf{r}$ and times $t$, which satisfy

$$\omega t - \mathbf{k}\cdot\mathbf{r} = \text{const}, \tag{1.4-12}$$

where the constant is arbitrary and determines the field value. Equation (1.4-12) determines a plane normal to the wave vector $\mathbf{k}$ at any instant $t$. This plane is called a *surface of constant phase*. The surfaces of constant phases are often referred to as *wavefronts*. The electromagnetic wave represented by Eq. (1.4-10) is called a plane wave because all the wavefronts are planar. It is easily seen that the surfaces of constant phase travel in the direction of $\mathbf{k}$ with a velocity whose magnitude is

$$v = \frac{\omega}{k}. \tag{1.4-13}$$

This is the phase velocity of the wave. We let $t = 0$ and examine the spatial variation, the separation between two neighboring field peaks, that is, the wavelength is

$$\lambda' = \frac{2\pi}{k} = 2\pi\frac{v}{\omega}. \tag{1.4-14}$$

where the prime indicates the wavelength of light inside the medium. In optics, $\lambda$ is reserved for the wavelength of light in a vacuum.